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ORTHODOX SEMIGROUPS ON WHICH GREEN'S RELATIONS ARE
COMPATIBLE, I. H-COMPATIBLE ORTHODOX SEMIGROUPS

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A semigroup S is said to be H $[L, R, D]$ -compatible if the Green's H $[L, R, D]$ -relation on S is a congruence. A strictly inversive semigroup introduced by the previous paper [5] of the author is of course H -compatible, and the structure of strictly inversive semigroups has been clarified by [5]. As a generalization of [5], the structure of H -compatible orthodox semigroups is studied in this paper. Further, we investigate L $[R]$ -compatible orthodox semigroups.

For an orthodox semigroup S , it is easily verified that the following conditions are equivalent:

Let E_S be the band of idempotents of S .

- (C1) If $e, f \in E_S$, $f \leq e$, $x \in S$ and $xx^* = x^*x = e$ (where x^* denotes an inverse of x) then $xf = fx$. (Note. $f \leq e$ means $ef = fe = f$.)
- (C2) If $e, f \in E_S$ and $f \leq e$, then $S(e) \cup S(f)$ (where $S(h)$ denotes the maximal subgroup of S containing h for each $h \in E_S$) is a subsemigroup of S .
- (C3) The union G_S of all maximal subgroups of S is a strictly inversive semigroup (that is, a (orthodox) band of groups; see [5]).
- (C4) The Green's H -relation (see [2]) on S is compatible (hence, it is a congruence; see [1]).

If S satisfies one of these conditions, then S is said to be H -compatible. If every maximal subgroup (that is, H -class containing an idempotent) of an orthodox semigroup A consists of a single element, then A is of course H -compatible. In this case, A is particularly called an H -degenerated orthodox semi-

group (abbrev., an H-d.o. semigroup). In this paper, as a generalization of the previous paper [5], we shall investigate the structure of H-compatible orthodox semigroups. The complete proofs are omitted and will be given in detail by [9].

§ 1. Spined product decomposition.

Let S be an H-compatible orthodox semigroup. Since G_S is a strictly inversive semigroup, there exist a semilattice Λ and a rectangular subgroup S_λ of G_S for each $\lambda \in \Lambda$ such that

$$G_S = \Sigma \{ S_\lambda : \lambda \in \Lambda \} \text{ (disjoint sum), and}$$

$$S_\alpha S_\beta \subset S_{\alpha\beta} \text{ for } \alpha, \beta \in \Lambda.$$

That is, G_S is decomposed into a semilattice Λ of rectangular groups S_λ . Further, such a decomposition of G_S is uniquely determined and is called the structure decomposition (see [5]); and this decomposition is denoted by $G_S \sim \Sigma \{ S_\lambda : \lambda \in \Lambda \}$.

Now, define a relation σ_S on S as follows:

(C5) $a \sigma_S b$ if and only if for any $e \in E_S^1$ (where E_S^1 denotes the adjunction of an identity 1 to E_S) there exist $\alpha, \beta \in \Lambda$ such that both aea^* and beb^* are contained in S_α and both a^*ea and b^*eb are contained in S_β .

Then,

Theorem 1. (1) σ_S is the finest H-degenerated inverse semigroup (abbrev., an H-d.i. semigroup) congruence on S ,

(2) for the natural homomorphism $\phi: S \rightarrow S/\sigma_S$, $\text{Ker } \phi = \{ \bar{e} \phi^{-1} : \bar{e} \in E_{S/\sigma_S} \} = \{ S_\lambda : \lambda \in \Lambda \}$, where \bar{e} is the σ_S -class containing e , and $\bigcup \text{Ker } \phi = G_S$,

(3) for the congruence ν_ϕ on S induced by ϕ , the restriction $\nu_\phi|_{G_S}$

gives the structure decomposition of G_S .

Next, we extend the concept of spined products introduced by [5] (see also [6]) from strictly inversive semigroups to H-compatible orthodox semigroups. Let $\phi_1: A \rightarrow \Gamma$ and $\phi_2: B \rightarrow \Gamma$ be homomorphisms of H-compatible orthodox semigroups A and B onto an H-d.i. semigroup Γ such that the congruences $\vee_{\phi_1}, \vee_{\phi_2}$ induced by ϕ_1, ϕ_2 give the finest H-d.i. semigroup decompositions of A, B respectively. Put $\gamma\phi_1^{-1} = A_\gamma, \gamma\phi_2^{-1} = B_\gamma$ for $\gamma \in \Gamma$. Define multiplication in $C = \bigcup \{A_\gamma \times B_\gamma : \gamma \in \Gamma\}$ as follows:

$$(a_\alpha, b_\alpha)(c_\beta, d_\beta) = (a_\alpha c_\beta, b_\alpha d_\beta)$$

for $(a_\alpha, b_\alpha) \in A_\alpha \times B_\alpha$ and $(c_\beta, d_\beta) \in A_\beta \times B_\beta$.

Then, C is also an H-compatible orthodox semigroup. This C is called the spined product of A and B with respect to (Γ, ϕ_1, ϕ_2) , and denoted by $(A \bowtie B; \Gamma, \phi_1, \phi_2)$.

Under this definition, we have the following:

Lemma 2. Let S be an H-compatible orthodox semigroup, and σ_S the finest H-d.i. semigroup congruence on S . If H-compatible orthodox semigroup congruences σ_1 and σ_2 on S satisfy

- (1) $\sigma_1 \subset \sigma_S$ and $\sigma_2 \subset \sigma_S$, and σ_1, σ_2 are permutable,
- (2) $\sigma_1 \wedge \sigma_2 = \iota_S$ (the identity congruence on S), and
- (3) $\sigma_1 \vee \sigma_2 = \sigma_S$ (where $\sigma_1 \vee \sigma_2$ denotes the least congruence on S containing both σ_1 and σ_2),

then S is isomorphic to $(S/\sigma_1 \bowtie S/\sigma_2; S/\sigma_S, \phi_1, \phi_2)$, where ϕ_1, ϕ_2 are the homomorphisms of $S/\sigma_1, S/\sigma_2$ to S/σ_S respectively defined by $(x\sigma_1)\phi_1 = x\sigma_S$ and $(x\sigma_2)\phi_2 = x\sigma_S$.

Now, define two relations σ_1, σ_2 on an H-compatible orthodox

semigroup S as follows:

- (1) $a \sigma_1 b$ if and only if $a \sigma_S b$ and $ab^*, b^*a \in E_S$ for some inverse b^* of b ,
- (2) $a \sigma_2 b$ if and only if $a \sigma_S b$ and
 $aea^*beb^* = beb^*, beb^*aea^* = aea^*$ for all $e \in E_S^1$ and for some inverses a^* and b^* ;
 $a^*eab^*eb = a^*ea, b^*eba^*ea = b^*eb$ for all $e \in E_S^1$ and for some inverses a^* and b^* .

Then, these σ_1, σ_2 satisfy the conditions (1), (2) and (3) of Lemma 2. Hence, S is isomorphic to $(S/\sigma_1 \bowtie S/\sigma_2; S/\sigma_S, \phi_1, \phi_2)$, where ϕ_1, ϕ_2 are the homomorphisms of $S/\sigma_1, S/\sigma_2$ to S/σ_S respectively defined by $(x\sigma_1)\phi_1 = x\sigma_S$ and $(x\sigma_2)\phi_2 = x\sigma_S$. In this case, it is easily seen that S/σ_1 and S/σ_2 are an H -compatible inverse semigroup and an H -d.o. semigroup respectively.

Thus, we have the following theorem.

Theorem 3. An orthodox semigroup S is H -compatible if and only if $S \cong (A \bowtie B; \Gamma, \phi_1, \phi_2)$, where A is an H -compatible inverse semigroup, B an H -d.o. semigroup, Γ an H -d.i. semigroup, and ϕ_1, ϕ_2 homomorphisms of A, B onto Γ respectively.

§ 2. Construction theorems.

The concept of a regular extension of an inversive semigroup [a band] by an inverse semigroup has been introduced by [8] [by [7]] as follows:

If a regular semigroup R contains an inversive semigroup [a band] T as its subsemigroup and if there exists a homomorphism ϕ of R onto an inverse semigroup Γ such that

- (1) $\bigcup \text{Ker } \phi \equiv \bigcup \{\lambda\phi^{-1} : \lambda \in E_\Gamma\} = T$, and

(2) the structure decomposition of T is $T \sim \Sigma\{\lambda\phi^{-1}: \lambda \in E_\Gamma\}$, then R is called a regular extension of T by Γ .

Further, the previous paper [8] [[7]] has given a description of all possible regular extensions of T by Γ for a given inversive semigroup [band] T and for a given inverse semigroup Γ .

Since every H-d.o. semigroup B can be obtained as a regular extension of a band by an H-d.i. semigroup (see [7]), the construction of H-compatible orthodox semigroups is reduced to the following problems:

- I. Construction of H-d.i. semigroups.
- II. Construction of H-compatible inverse semigroups.

Now, we shall consider about these two problems.

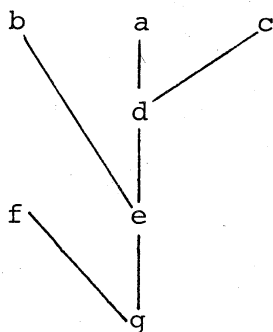
I. Let E be a semilattice, and put $\{(e,f): eE \cong fE\} = \Delta$. Let Ω be a subset of Δ and $\phi_{(e,f)}$ an isomorphism of eE onto fE for each $(e,f) \in \Omega$. Assume that Ω and the family $F_\Omega(E) = \{\phi_{(e,f)}: (e,f) \in \Omega\}$ satisfy the following:

- (C6) (1) $(e,e) \in \Omega$ for all $e \in E$,
- (2) $(e,f) \in \Omega$ implies $(f,e) \in \Omega$,
- (3) $\phi_{(e,e)}$ is the identity mapping on eE for each $e \in E$,
- (4) $(e,f), (h,t) \in \Omega$ implies $((fh)\phi_{(f,e)}, (fh)\phi_{(h,t)}) \in \Omega$
 and $\phi_{((fh)\phi_{(f,e)}, (fh)\phi_{(h,t)})} = \phi_{(e,f)}\phi_{(h,t)} \mid (fh)\phi_{(f,e)}^E$.

For example, consider the case where E satisfies the following condition:

- (C7) For any $e \in E$, eE is a totally ordered set (with respect to the usual ordering); and when $e, f \in E$, $e \neq f$ and $eE \cong fE$, an isomorphism θ of eE onto fE is unique. (For example, see the

diagram below.)



In this case, select Ω and $\phi_{(e,f)}$ as follows:

- (1) $\Omega = \Delta$,
- (2) $\phi_{(e,e)}$ is the identity mapping on eE for each $e \in E$; and $\phi_{(e,f)}$ is the unique isomorphism of eE onto fE for each $(e,f) \in \Delta$ such that $e \neq f$.

Then, the system $\{\Delta, \phi_{(e,f)}\}$ satisfies (1)-(4) of (C6).

Now, $F_\Omega(E)$ above is an H-d.i. subsemigroup of the symmetric inverse semigroup \mathcal{I}_E on E . This $F_\Omega(E)$ is denoted by the symbol $HD(E; \Omega, \phi_{(e,f)})$.

Then, we have the following result:

Theorem 4. Any H-d.i. semigroup A is isomorphic to some $HD(E; \Omega, \phi_{(e,f)})$. In this case, the semilattice E_A can be selected as E . In particular, if E_A satisfies (C7) then A is isomorphic to a full regular subsemigroup of $HD(E_A; \Delta, \phi_{(e,f)})$ (where $\Delta = \{(e,f): e, f \in E_A, eE \cong fE\}$; $\phi_{(e,e)}$ = the identity mapping on eE for each $e \in E_A$; and $\phi_{(e,f)}$ = the unique isomorphism of eE onto fE for each $(e,f) \in \Delta$ such that $e \neq f$).

Remark. This result can be obtained by slightly modifying the results given by [3].

II. It is easily verified that an inverse semigroup is H-compatible if and only if it is a regular extension of a weakly C-inversive semigroup (that is, a semilattice of groups; see [6])

by an H-d.i. semigroup. Hence, hereafter we consider the problem of constructing all possible regular extensions of G by Γ for any given H-d.i. semigroup Γ having Λ as the semilattice of its idempotents and for any given weakly C-inversive semigroup G having $G \sim \Sigma \{ G_\lambda : \lambda \in \Lambda \}$ as its structure decomposition (that is, G is a semilattice Λ of groups G_λ).

Let e_λ be the identity element of G_λ .

The regular extensions of G by Γ are obtained by slightly modifying the concept of a Schreier extension in the group theory.

Let $\Psi: \Gamma \rightarrow \text{End}(G)$ (the semigroup of endomorphisms on G) and $\phi: \Gamma \times \Gamma \rightarrow G$ be mappings such that each $\bar{\gamma} = \gamma\Psi$ maps G_λ into $G_{\gamma\lambda(\gamma\lambda)^*}$, especially maps $G_{\gamma^*\gamma}$ onto $G_{\gamma\gamma^*}$, and each $C(\gamma, \tau) = (\gamma, \tau)\phi$ is an element of $G_{\gamma\tau(\gamma\tau)^*}$. If $\{\bar{\gamma} : \gamma \in \Gamma\}$ and $\{C(\gamma, \tau) : (\gamma, \tau) \in \Gamma \times \Gamma\}$ satisfy

$$(C8) \quad (1) \quad C(\lambda, \delta) = e_{\lambda\delta} \text{ for } \lambda, \delta \in E_\Gamma, \text{ and } C(\gamma\gamma^*, \gamma) = C(\gamma, \gamma^*\gamma) = e_{\gamma\gamma^*} \text{ for each } \gamma \in \Gamma,$$

$$(2) \quad C(\tau, \xi) \bar{\gamma} C(\gamma, \tau\xi) = C(\gamma, \tau) C(\gamma\tau, \xi), \text{ where } x \bar{\gamma} = x\bar{\gamma},$$

$$(3) \quad \bar{\gamma} \bar{\tau} = \overline{\tau\gamma C(\tau, \gamma)} \text{ (where, for each } a \in G, \bar{a} \text{ denotes the endomorphism on } G \text{ defined by } z \bar{a} = aza^*, z \in G),$$

$$(4) \quad e_\lambda b = b \bar{\lambda} \text{ for } \lambda, \delta \in E_\Gamma \text{ and } b \in G_\delta,$$

then the system $\{\bar{\gamma}, C(\gamma, \xi)\}$ is called a (normalized) factor set of G belonging to Γ .

Now, under this definition we have the following theorem:

Theorem 5. Let G , Γ and Λ be as above. Let $\{\bar{\gamma}, C(\gamma, \xi)\}$ be a factor set of G belonging to Γ . If multiplication is defined in $G \rtimes \Gamma = \{(a, \gamma) : a \in G_{\gamma\gamma^*}, \gamma \in \Gamma\}$ by

$$(a, \gamma)(b, \tau) = (a\bar{\gamma}C(\gamma, \tau), \gamma\tau),$$

then

- (1) $\underline{G} = \{ (a, \lambda) : a \in G_\lambda, \lambda \in E_\Gamma \}$ is a weakly C-inversive semigroup which is isomorphic to G , and
- (2) $G \rtimes \Gamma$ is a regular extension of \underline{G} by Γ .

Further, every regular extension of G by Γ is obtained in this fashion, up to isomorphism.

§ 3. L-compatible regular semigroups.

A semigroup S is said to be L-compatible [R-compatible] if the Green's L-relation [R-relation] on S is compatible (that is, a congruence). Firstly, we have the following theorem.

Theorem 6. A regular semigroup S is L-compatible if and only if it is a right regular band of left groups. Of course, in this case S is a union of groups.

A band B is called a regular [left regular, right regular] band if B satisfies the identity $xyxzx = xyzx$ [$xyx = xy$, $yx = yx$] (see [4]). If a semigroup S is an orthodox semigroup and is a right regular band of left groups, then S is called an orthodox right regular band of left groups. An orthodox band of groups, an orthodox regular band of groups, etc. are similarly defined.

Under these definitions and by using Theorem 6, we have the following result:

Corollary 7. A semigroup is an L-compatible orthodox semigroup if and only if it is an orthodox right regular band of left groups.

By using Theorem 6 and its dual result, the following is also obtained:

Theorem 8. A regular semigroup is both L-compatible and R-compatible if and only if it is a regular band of groups.

Corollary 9. A semigroup is a both L- and R-compatible orthodox semigroup if and only if it is both a semilattice of rectangular groups and a regular band of groups.

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